A GOAL-CONSTRAINT FORMULATION FOR MULTI-ITEM INVENTORY SYSTEMS

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THESIS

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September 1970

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Submitted in partial fulfillment of the
requirements for the degree of

MASTER OF SCIENCE IN OPERATIONS RESEARCH

from the

NAVAL POSTGRADUATE SCHOOL
September 1970

ABSTRACT

Variations on the traditional cost minimization of continuous review formulation are investigated in an effort to improve service as measured in terms of time-weighted shortages per unit time. It is proposed that the minimization of time-weighted shortages per unit time will improve service in current Navy Supply Operations. Various models are presented, without reliance upon unknown parameters such as order cost and carrying cost, with necessary conditions and solution algorithms.

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ACKNOWLEDGEMENT

I wish to express my appreciation to Professor David A. Schrady whose valuable criticisms and suggestions contributed substantially to the completion of this paper.

I. INTRODUCTION

Inventories exist to provide service to customers by satisfying their demands from on-hand material. It follows, then, that a reasonable objective of inventory management is the maximization of service provided which is achieved by minimizing stockouts. In particular, total time-weighted shortages is thought to be the desired objective.

In pursuing this objective, the manager of a realistically large, multi-item inventory system has a number of constraints imposed on his "when to buy and how much to buy" decisions. The stock points of the Navy Supply System have investment and reorder workload constraints which are real and binding.

The classic variable cost minimization formulation is the most used method for solving this inventory problem. Multi-item problems are usually solved by assuming that they can be dealt with as a series of independent single item problem. In the presence of binding constraints on a population of items this approach is not applicable. Additionally the cost minimization formulation requires the estimation of cost parameters which are arbitrary or at least very difficult to estimate.

As a consequence of this argument, a series of models are formulated for multi-item policies subject to investment and reorder workload constraints. These models do not employ the standard ordering, shortage costs. This approach was suggested by A. P. Tully [1] .

In the next section, the problem formulation and the general models are developed. Section III develops the single item model as preparatory to studying multi-item case. Section IV presents a simplified multi-item

formulation in whicn only the items reorder points are decision variables. The general multi-item continuous review model is developed in Section V. In the final section, we summarize and state some tentative conclusions.

II. FORMULATION

It is desired to formulate inventory decision rules for multi-item inventories subject to specific constraints. The inventory decision rules will be of the reorder point - reorder quantity, continuous review type.

As suggested by the introduction, the formulation to be used involves the minimization of total time-weighted shortages subject to:

(1) Total average investments costs $\leq$ investment limit; and

(2) total number of orders $\leq$ reorder workload constraints.

Note that such a formulation would not be based on minimizing variable costs.

The specific form of the model depends upon the assumption about the item demand characteristics and expressions for the total average on-hand inventory level and total number of buys per unit time. The first assumption is the distribution of lead time demand is normal (μ_i, σ_i^2) for all items.

The following notation is used throughout the paper. For the i-th item let;

C_i = item unit cost in dollars;

λ_i = mean demand per unit time in units;

μ_i = mean lead time demand in units;

σ_i = Standard deviation of lead time demand in units;

$\Phi(r_i)$ = probability that lead time demand exceeds r;

r_i = reorder point; and

Q_i = reorder quantity.

Also let;

K_1 = investment limit; and

K_2 = reorder workload constraint.

With a continuous review inventory policy an order is placed after the demand of Q units. It follows then that the average number of orders per unit time is $\frac{\lambda}{Q}$. For a multi-item inventory with N items, the total expected number of orders placed per unit time is

$$\sum_{i=1}^{N} \frac{\lambda_i}{Q_i} . \tag{II.1}$$

Total inventory investment is the priced-out value of the total expected on-hand inventory. As shown by Hadley and Whitin [2] for continuous review the expected on-hand quantity, E(OH) is given by

$$E(OH) = r + \frac{Q}{2} - \mu + B\ (Q,r),$$

where B(Q, r) is the expression for the expected shortages at any point of time. If lead time demand is normally distributed it can be shown [2] that

$$B(Q,\ r) = \frac{1}{Q}\ \left[\beta(r) - \beta\ (r+Q)\ \right], \tag{II.2}$$

where

$$\beta(r) = \frac{1}{2}\left[\sigma^2 + (r-\mu)^2\right] \Phi\left(\frac{r-\mu}{\sigma}\right) - \frac{\sigma}{2}\left(\frac{r-\mu}{\sigma}\right)\phi\left(\frac{r-\mu}{\sigma}\right);$$

$$\phi(X) = \frac{1}{\sqrt{2\pi}}\ e^{-\frac{1}{2}X^2} \quad ; \text{ and}$$

$$\Phi(r) = \int_r^{\infty} \phi\,(X)\ d\,X.$$

The expected on-hand quantity expression can be simplified by omitting the B(Q,r) term, and this approximation is reasonable if the risk of stock out is not too large. This assumption is employed throughout the thesis.

With this assumption, the total inventory investment is then given by the expression

$$\sum_{i=1}^{N} Ci \left(ri + \frac{Qi}{2} - \mu i \right). \tag{II.3}$$

The expected number of backorders at any time may be explicitly determined from the steady state probability distribution for negative net inventory levels. Hadley and Whitin [2] used this approach and showed that when lead time demand is normal, the time-weighted shortages expression is given by Eq (II.2).

If the risk of stockout is small, then the expression for the time-weighted shortages can be simplified by ignoring the $\beta(r+Q)$ term, which yields expected time-weighted units short per unit time for the ith item as:

$$\frac{1}{Qi} \beta(ri) \tag{II.4}$$

The objective can now be stated as the minimization of the time-weighted shortages for the entire inventory, and the formulation is:

$$\min Z = \sum_{i=1}^{N} \frac{1}{Qi} \beta(ri) \tag{II.5}$$

subject to:

$$\sum_{i=1}^{N} Ci\left(ri + \frac{Qi}{2} - \mu i \right) \leq K_1 , \tag{II.6}$$

$$\sum_{i=1}^{N} \frac{\lambda i}{Qi} \leq K_2 , \tag{II.7}$$

$Qi > 0$, and r_i unrestricted.

Note that the investment constraint will always be active, given the objective function used, but that reorder constraint may or may not be active in a given problem.

III. SINGLE ITEM B(Q,r) MODEL

The basic continuous review formulation for one item with normal (μ, σ^2) lead time demand can be written as

$$\text{min:} \quad \frac{1}{Q} \quad \beta(r) \tag{III.1}$$

$$\text{subject to:} \quad C\,(r + \frac{Q}{2} - \mu) \leq K_1 \,, \tag{III.2}$$

$$\frac{\lambda}{Q} \leq K_2 \,, \tag{III.3}$$

$Q > 0$, and r unrestricted.

A. NECESSARY CONDITIONS

To solve the single item continuous review model, set up the Lagrangean function

$$L(Q, r, \eta, \theta) = \frac{1}{Q} \beta(r) + \eta\left[C(r + \frac{Q}{2} - \mu) - K_1\right] + \theta\left[\frac{\lambda}{Q} - K_2\right] \tag{III.4}$$

Taking the partial derivatives $\frac{\partial L}{\partial Q}$, $\frac{\partial L}{\partial r}$, $\frac{\partial L}{\partial \eta}$, and $\frac{\partial L}{\partial \theta}$, and setting the resulting expressions equal to zero yield;

$$\frac{\partial L}{\partial Q} = -\frac{1}{Q^2} \beta(r) + \frac{\eta C}{2} - \frac{\theta \lambda}{Q^2} = 0 \quad , \tag{III.5}$$

$$\frac{\partial L}{\partial r} = \frac{1}{Q}\left[(r-u)\Phi\left(\frac{r-\mu}{\sigma}\right) - \sigma\phi\left(\frac{r-\mu}{\sigma}\right)\right] + \eta C = 0 \,, \tag{III.6}$$

$$\frac{\partial L}{\partial \eta} = C(r + \frac{Q}{2} - \mu) - K_1 = 0 \,, \text{ and} \tag{III.7}$$

$$\frac{\partial L}{\partial \theta} = \frac{\lambda}{Q} - K_2 = 0 \,. \tag{III.8}$$

These equations may be rewritten as

$$Q^2 = \frac{2\left[\beta(r) + \theta\lambda\right]}{\eta C}, \qquad \text{(III.9)}$$

$$\sigma\,\phi\left(\frac{r-\mu}{\sigma}\right) - (r-\mu)\,\Phi\left(\frac{r-\mu}{\sigma}\right) = \eta CQ, \qquad \text{(III.10)}$$

$$C\left(r + \frac{Q}{2} - \mu\right) = K_1, \qquad \text{(III.11)}$$

and $$\frac{\lambda}{Q} = K_2 \,. \qquad \text{(III.12)}$$

These are the necessary conditions for solution.

B. ITERATIVE SCHEME

If the reorder constraint is active, the reorder quantity is determined from the equation (III.12)

$$\frac{\lambda}{Q} = K_2 \quad \text{which implies:}$$

$$Q = \frac{\lambda}{K_2} \,. \qquad \text{(III.13)}$$

Note that in this case, equation (III.11) can be solved for r yielding

$$r = \frac{K_1}{C} - \frac{\lambda}{2K_2} + \mu \,. \qquad \text{(III.14)}$$

Hence (Q^*, r^*) are uniquely determined from equations (III.13) and (III.14).

If the reorder constraint is not active, the reorder quantity is determined along the line which is equation (III.11). It is observed that the boundary conditions are as follows: when $r = 0$, $Q = 2\left(\frac{K_1}{C} + \mu\right)$, and when $Q = 0$, $r = \frac{K_1}{C} + \mu$. Plotting these conditions, Figure 1 is obtained.

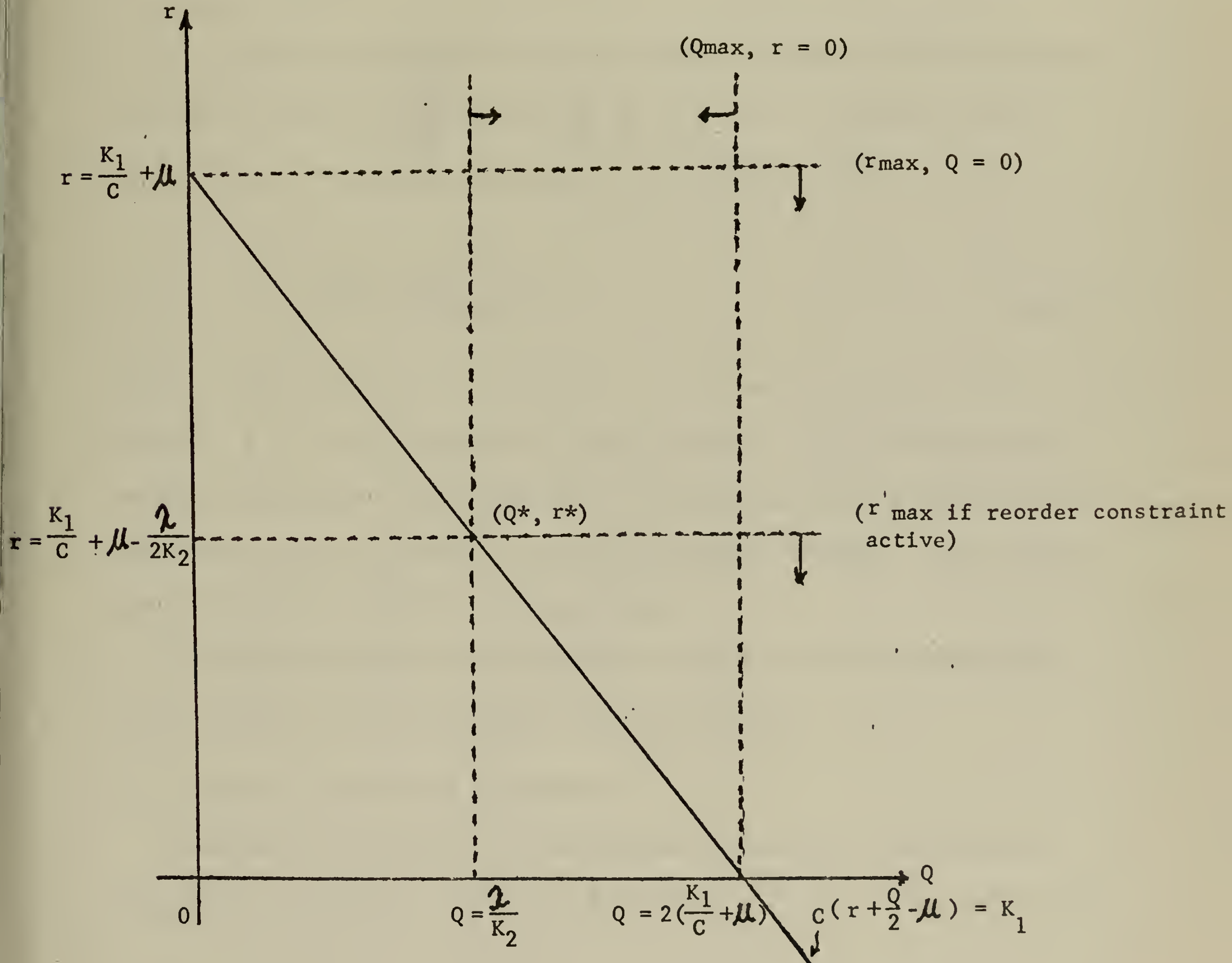

Figure 1. Boundary Conditions for Single Item Case.

It is realized that if $\frac{\lambda}{K_2}$ is greater than $2(\frac{K_1}{C} + \mu)$, then the investment and reorder constraints are mutually inconsistent (infeasible). If all solutions lie on the line $C(r + \frac{Q}{2} - \mu) = K_1$ for $Q \geq \frac{\lambda}{K_2}$ and $r \leq \frac{K_1}{C} + \mu - \frac{\lambda}{2K_2}$, then pick up a Q, solve for from the equation (III.11), evaluate $\frac{\beta(r)}{Q}$ and continue to iterate on Q until $\frac{\beta(r)}{Q}$ minimum is found. Here parameterizing K_1 will give shortages as a function of the investment limit.

An alternative approach would be to use a Lagrangean function with the boundary conditions $\left[\frac{\lambda}{K_2} \leq Q \leq 2(\frac{K_1}{C} + \mu)\right]$ and the equation (III.9) with $\theta = 0$. From these equations, it is determined that

$$\eta \geq \frac{\beta(r)}{2(\frac{K_1}{C} + \mu)^2 C} \qquad \text{(III.15)}$$

Hence a double iteration is required as the function of η, Q, and r ; i.e., iterate Q, r for a given value of the multiplier. The solution is the minimum time-weighted shortages for the multiplier used; minimum for the investment level (K_1) imputed to that multiplier value η. Then a suggested procedure would be a binary search.

Plotting shortages vs. multipliers on the Q, r and shortages planes, the different level of shortages will be obtained.

C. EXAMPLE OF THE SINGLE ITEM MODEL

Consider an item with its distribution of lead time demand normal (μ_1, σ^2). Let $\mu = 125$, $\sigma^2 = 25$, $\lambda = 500$, $C = 12$, $K_1 = 720$, and $K_2 = 8$.

The problem is

$$\text{minimize} \quad Z = \frac{1}{Q}\,\beta(r)$$

$$\text{subject to:} \quad C\left(r + \frac{Q}{2} - \mu\right) \lesssim K_1,$$

$$\frac{\lambda}{Q} \lesssim K_2,$$

$Q > 0$, and r unrestricted.

From $\frac{\lambda}{Q} \lesssim K_2$, it is determined that $Q \gtrsim \frac{500}{8} = 62.5$. Hence if the reorder constraint is active, $Q^* = 62.5$, r is obtained from (III.11) as $r^* = 144$, and $Z^* = .606$.

If the reorder constraint is not active, then a double iteration is required as the function η, Q, and r. The solution is the minimum time-weighted shortages for the multiplier used. Iterating on Q, and ignoring the reorder constraint, produces the following results:

Q	r	Time-weighted shortages per unit time	Expected number of units short per unit time	
10	170	0.3198	17.8446	
18	166	0.027	14.5262	*
36	157	0.333	16.2599	
50	150	0.471	20.8288	
62	144	0.606	25.0432	
76	137	0.904	32.9660	
90	130	1.247	42.6242	
105	122	1.741	53.6828	

Table 1. Time-weighted Shortages and Expected Number of Units Short for Single Item Case.

Note: Two approaches seem to follow together.

IV. SIMPLIFIED MULTI-ITEM B(Q,r) MODEL

The basic continuous review formulation for the multi-item problem was given as:

$$\text{minimize} \quad Z = \sum_{i=1}^{N} \frac{\beta(ri)}{Qi} \tag{IV.1}$$

$$\text{subject to:} \quad \sum_{i=1}^{N} Ci(\, ri + \frac{Qi}{2} - \mu i) \leq K_1 \,, \tag{IV.2}$$

$$\sum_{i=1}^{N} \frac{\lambda i}{Qi} \leq K_2 \,, \tag{IV.3}$$

$Qi > 0$, and r_i unrestricted.

Suppose the order quantities are fixed by some other criterion. Specifically the assumption is made that order quantities are determined from the equation $Q_i = G\sqrt{\frac{\lambda i}{Ci}}$, (IV.4)

where G is a constant, which is assumed to be the same for all items.

From the reorder constraint (IV.3) and the assumption (IV.4) it follows that

$$\sum_{i=1}^{N} \frac{\sqrt{\lambda iCi}}{G} = K_2 \,, \tag{IV.5}$$

or

$$G = \frac{\sum_{i=1}^{N} \sqrt{\lambda iCi}}{K_2} \,.$$

The determination of G then fixes the order quantities from equation (IV.4) and eliminates one set of decision variables from the problem, i.e.,

$$Q_i = \frac{\sum_{i=1}^{N} \sqrt{\lambda_i C_i}}{K_2} \sqrt{\frac{\lambda_i}{C_i}} \quad . \tag{IV.6}$$

Substituting (IV.4) into equation (IV.2), it follows that

$$\sum_{i=1}^{N} C_i \left(r_i + \frac{G}{2} \sqrt{\frac{\lambda_i}{C_i}} - \mu_i \right) \leq K_1 . \tag{IV.7}$$

Equation (IV.7) can be reduced to the form

$$\sum_{i=1}^{N} C_i \, r_i \leq K_1 - \frac{G}{2} \sum_{i=1}^{N} \sqrt{\lambda_i C_i} + \sum_{i=1}^{N} C_i \mu_i = K_1' \ . \tag{IV.8}$$

Now the multi-item B(Q, r) Model with fixed Q's can be written as

$$\text{minimize } Z = \sum_{i=1}^{N} \frac{\beta(r_i)}{Q_i}$$

$$\text{subject to:} \quad \sum_{i=1}^{N} C_i r_i \leq K_1' \ ,$$

and r_i unrestricted,

$$\text{where} \quad Q_i = \frac{\sum_{i=1}^{N} \sqrt{\lambda_i C_i}}{2K_2} \sqrt{\frac{\lambda_i}{C_i}} = G \sqrt{\frac{\lambda_i}{C_i}} \ .$$

A. NECESSARY CONDITIONS

To solve the simplified model, set up Lagrangean function

$$L(r_i, \eta) = \sum_{i=1}^{N} \frac{1}{G} \sqrt{\frac{C_i}{\lambda_i}} \, \beta(r_i) + \eta \left[\sum_{i=1}^{N} C_i r_i - K_1' \right] . \tag{IV.9}$$

Taking the partial derivatives with respect to decision variables and setting these expressions equal to zero yields

$$\frac{\partial L}{\partial r_i} = \frac{1}{G}\sqrt{\frac{C_i}{\lambda_i}}\left[(r_i-\mu_i)\,\Phi\left(\frac{r_i-\mu_i}{\sigma_i}\right) - \sigma_i\,\phi\left(\frac{r_i-\mu_i}{\sigma_i}\right)\right] + \eta C_i = 0, \text{ and}$$

$$\frac{\partial L}{\partial \eta} = \sum_{i=1}^{N} C_i r_i - K_1' = 0.$$

Thus the conditions for solution to the problem are

$$\eta = -\frac{1}{G\sqrt{C_i\lambda_i}}\left[(r_i-\mu_i)\,\Phi\left(\frac{r_i-\mu_i}{\sigma_i}\right) - \sigma_i\,\phi\left(\frac{r_i-\mu_i}{\sigma_i}\right)\right],$$

or

$$\eta = \frac{1}{G\sqrt{C_i\lambda_i}}\left[\sigma_i\,\phi\left(\frac{r_i-\mu_i}{\sigma_i}\right) - (r_i-\mu_i)\,\Phi\left(\frac{r_i-\mu_i}{\sigma_i}\right)\right], \qquad \text{(IV.10)}$$

and

$$\sum_{i=1}^{N} C_i r_i = K_1'. \qquad \text{(IV.11)}$$

B. NUMERICAL SOLUTION SCHEME

In general, these equations cannot be solved in closed form. A numerical solution procedure is suggested. Let us consider a numerical method of solving equations (IV.10) and (IV.11). Observing equation (IV.10), which is

$$\eta = \frac{1}{G\sqrt{C_i\lambda_i}}\left[\sigma_i\,\phi\left(\frac{r_i-\mu_i}{\sigma_i}\right) - (r_i-\mu_i)\,\Phi\left(\frac{r_i-\mu_i}{\sigma_i}\right)\right],$$

note that the right-hand side has a lower bound of 0 since $\eta \geqslant 0$ from $G > 0$, $C_i\lambda_i > 0$ and $\sigma_i\,\phi\left(\frac{r_i-\mu_i}{\sigma_i}\right) - (r_i-\mu_i)\,\Phi\left(\frac{r_i-\mu_i}{\sigma_i}\right)$ 0 for all r_i. Specifically $\eta = 0$ when r_i becomes infinite and $\sigma_i\,\phi\left(\frac{r_i-\mu_i}{\sigma_i}\right) - (r_i-\mu_i)\cdot\Phi\left(\frac{r_i-\mu_i}{\sigma_i}\right) = 0$, but $r_i = \infty$ violates the investment constraint.

Hence for the initial value of η it is reasonable to start with $r_i = 0$, then this implies

$$\eta = \frac{1}{G\sqrt{c_i \lambda_i}} \left[\sigma_i \phi(-\frac{\mu_i}{\sigma_i}) + \mu_i \Phi(-\frac{\mu_i}{\sigma_i}) \right] \text{ for all } i,$$

or

$$\eta \leq \frac{1}{G} \min \left[\frac{1}{\sqrt{c_1\lambda_1}} \left\{ \sigma_1 \phi(-\frac{\mu_1}{\sigma_1}) + \mu_1 \Phi(-\frac{\mu_1}{\sigma_1}) \right\}, \frac{1}{\sqrt{c_2\lambda_2}} \left\{ \sigma_2 \phi(-\frac{\mu_2}{\sigma_2}) + \mu_2 \Phi(-\frac{\mu_2}{\sigma_2}) \right\}, \dots \right].$$

Let $$\delta = \frac{1}{G} \min \frac{1}{\sqrt{c_i\lambda_i}} \left[\sigma_i \phi(-\frac{\mu_i}{\sigma_i}) + \mu_i \Phi(-\frac{\mu_i}{\sigma_i}) \right] \text{ for all } i;$$

$\eta = \delta$ implies that there is at least one r_i at its lower bound of zero. Hence $\eta = \frac{\delta}{2}$ is a convenient starting point. Then a suggested solution procedure would be to begin at $\eta = \frac{\delta}{2}$, solve equation (IV.10) for the r_i's and compute the value of constraint using equation (IV.12), which is

$$\sum_{i=1}^{N} c_i r_i = H. \qquad \text{(IV.12)}$$

A bisection search will be used again. If $H > K_1'$, increase η by $\frac{\delta}{4}$. If $H < K_1'$, decrease η by $\frac{\delta}{4}$. Recompute the r_i's and the value of constraint using equation (IV.12). If the increase (or decrease) of η has not caused the change of inequality sign, increase (or decrease) η by the same amount $\frac{\delta}{4}$. If the sign of inequality has changed, then reduce the increment to $\frac{\delta}{8}$ and increase (or decrease) η, solving for the r_i's at each value of η and computing the value of H until the sign of the inequality switches again. Continue until $H = K_1'$ or until H is within some tolerable limit of K_1'. This approach will converge to the optimal solution rapidly.

From the Kuhn-Tucker theorem [3], if we have a convex objective function and a convex constraint region, the necessary conditions are also sufficient. Since the constraint under consideration is linear in r, the region is convex. To show $Z(r_i)$ is convex, consider the equation of the expected time-weighted shortages,

$$Z_i = \frac{1}{G}\sqrt{\frac{C_i}{\lambda_i}}\;\beta(r_i).$$

Now if $\dfrac{\partial^2 Z_i}{\partial r_i^2} \geqslant 0$ for all r_i, then Z_i is convex. Taking partial derivatives,

$$\frac{\partial Z_i}{\partial r_i} = \frac{1}{G}\sqrt{\frac{C_i}{\lambda_i}}\left[(r_i-\mu_i)\,\Phi\left(\frac{r_i-\mu_i}{\sigma_i}\right) - \sigma_i\,\phi\left(\frac{r_i-\mu_i}{\sigma_i}\right)\right] < 0,$$

and

$$\frac{\partial^2 Z_i}{\partial r_i^2} = \frac{1}{G}\sqrt{\frac{C_i}{\lambda_i}}\;\Phi\left(\frac{r_i-\mu_i}{\sigma_i}\right). \qquad \text{(IV.13)}$$

Equation (IV.13) will always be greater than or equal to zero. Under these conditions Z_i is convex. It follows that Z is convex since it is the sum of convex functions.

C. EXAMPLE OF THE SIMPLIFIED MULTI-ITEM B(Q,r) MODEL

Let us consider an inventory of three items. It is assumed that the distribution of lead time demand is normal with mean μ_i and variance σ_i^2 for the i^{th} item. Let the item data be as follows:

	Item 1	Item 2	Item 3
λ_i	1000	1500	2000
C_i	1	10	20
μ_i	100	200	300
σ_i^2	100	100	200

, and let

K_1 = \$8,000 and K_2 = 15.

Now the problem is

$$\text{minimize } Z = \frac{1}{G} \sum_{i=1}^{3} \sqrt{\frac{Ci}{\lambda i}} \; \beta\,(ri)$$

subject to:

$$\sum_{i=1}^{3} Ci\, ri \leq K_1' = K_1 - \frac{G}{2} \sum_{i=1}^{3} \sqrt{\lambda i Ci} + \sum_{i=1}^{3} Ci\, \mu i \,,$$

where

$$G = \frac{\sum_{i=1}^{3} \sqrt{\lambda i Ci}}{K_2} = 23.6065.$$

order quantities are determined from equation (IV.4),

$$Q1 = 746.5022,$$

$$Q2 = 289.1190, \text{ and}$$

$$Q3 = 236.0648\ .$$

With these order quantities, the problem now can be stated as

$$\text{minimize } Z = \sum_{i=1}^{3} \frac{1}{23.6065} \sqrt{\frac{Ci}{\lambda i}} \; \beta\,(ri)$$

subject to:

$$\sum_{i=1}^{3} Ci\, ri \leq 8000 + \sum_{i=1}^{3} Ci\mu i - \sum_{i=1}^{3} \frac{QiCi}{2} = 11{,}920.5078.$$

Notice that the solution to the above problem is that vector r such that

$$\eta = \frac{1}{G\sqrt{Ci\,ri}} \left[\sigma i\, \phi\left(\frac{ri - \mu i}{\sigma i}\right) - (ri - \mu i)\, \Phi\left(\frac{ri - \mu i}{\sigma i}\right)\right] \text{, for all i. (IV.10)}$$

For each value of η, there will be some $\{r\}$ for which equation (IV.10) is satisfied. However, having the convex objective function, there exists only one $\{r_i\}$ for all i such that equations (IV.10) and (IV.11) are satisfied at the same time. For the initial value of $\eta = \frac{\delta}{2}$, using $\delta = \frac{1}{G} \min \frac{1}{\sqrt{C_i \lambda_i}} \left[\sigma_i \phi(-\frac{\mu_i}{\sigma_i}) + \mu_i \Phi(-\frac{\mu_i}{\sigma_i}) \right]$ for all i when $r_i = 0$, $\eta_o = 0.0318$ will be used. Decreasing η from 0.0318, $\eta_F = 0.0055$ is obtained, which implies

$r_1 = 234.3750$,

$r_2 = 264.0625$, and

$r_3 = 453.1250$.

Checking the constraint, it is found that

$$\sum_{i=1}^{3} C_i r_i = 11937.5000,$$

which is within 0.2 percent level of K_1'.

The expression for the time-weighted shortage per item per unit time is

$$z_i = \frac{1}{G}\sqrt{\frac{C_i}{\lambda_i}} \left[\frac{1}{2} \left\{ \sigma_i^2 + (r_i - \mu_i)^2 \right\} \Phi(\frac{r_i - \mu_i}{\sigma_i}) - \frac{1}{2} \sigma_i (r_i - \mu_i) \phi(\frac{r_i - \mu_i}{\sigma_i}) \right].$$

The time-weighted shortage per unit time is computed as

$$z_1 = \frac{1}{23.6065} \sqrt{\frac{1}{1000}} \left[\frac{1}{2} \left\{ 100^2 + (240-100)^2 \right\} \Phi(\frac{140}{100}) - \frac{100}{2} (240-100) \phi(\frac{140}{100}) \right]$$

$= 0.2179$,

$$z_2 = \frac{1}{23.6065} \sqrt{\frac{10}{1500}} \left[\frac{1}{2} \left\{ 100^2 + (285-200)^2 \right\} \Phi(\frac{85}{100}) - \frac{100}{2} (285-200) \phi(\frac{85}{100}) \right]$$

$= 2.7634$, and

$$z_3 = \frac{1}{23.6065}\sqrt{\frac{20}{2000}}\left[\frac{1}{2}\left\{200^2 + (285-300)^2\right\}\Phi(-\frac{15}{200}) - \frac{200}{2}(285-300)\phi(-\frac{15}{200})\right]$$

$$= 10.5232 .$$

Total time-weighted shortages per unit time for the entire inventory are

$$z = \sum_{i=1}^{3} z_i = 13.5045.$$

V. GENERAL MULTI-ITEM B (Q, r) MODEL

Let us consider the completely general continuous review model for multi-item B(Q, r) Model which was:

minimize $$Z = \sum_{i=1}^{N} \frac{Wi}{Qi} \beta(ri),$$

subject to: $$\sum_{i=1}^{N} Ci\left(ri + \frac{Qi}{2} - \mu i\right) \leq K_1$$

$$\sum_{i=1}^{N} \frac{\lambda_i}{Qi} \leq K_2$$

$Qi > 0$, and r_i unrestricted,

where $$\beta(ri) = \frac{1}{2}\left[\sigma_i^2 + (ri - \mu i)^2\right] \Phi\left(\frac{ri-\mu i}{\sigma i}\right) - \frac{\sigma_i}{2}(ri - \mu i)\phi\left(\frac{ri-\mu i}{\sigma i}\right),$$

and Wi is a weighting factor.

A. NECESSARY CONDITIONS

Applying the Lagrange Multiplier technique, the following results will be obtained:

$$L(Qi, r_i, \eta, \theta) = \sum_{i=1}^{N} \frac{Wi}{Qi} \beta(ri) + \eta\left[\sum_{i=1}^{N} Ci\left(ri + \frac{Qi}{2} - \mu i\right) - K_1\right] + \theta\left[\sum_{i=1}^{N} \frac{\lambda i}{Qi} - K_2\right]; \quad (V.1)$$

$$\frac{\partial L}{\partial Qi} = -\frac{Wi}{Q_i^2} \beta(ri) + \frac{\eta Ci}{2} - \frac{\theta \lambda_i}{Q_i^2} = 0; \quad (V.2)$$

$$\frac{\partial L}{\partial ri} = \frac{\omega_i}{Qi}\left[(ri - \mu i)\,\Phi\left(\frac{ri-\mu i}{\sigma i}\right) - \sigma_i \phi\left(\frac{ri-\mu i}{\sigma i}\right)\right] + \eta Ci = 0; \quad (V.3)$$

$$\frac{\partial L}{\partial \eta} = \sum_{i=1}^{N} Ci\left(ri + \frac{Qi}{2} - \mu i\right) - K_1 = 0; \text{ and} \quad (V.4)$$

$$\frac{\partial L}{\partial \theta} = \sum_{i=1}^{N} \frac{\lambda_i}{Q_i} - K_2 \quad = \quad 0. \tag{V.5}$$

Then equations (V.1) and (V.2) can be solved for each multiplier yielding

$$\theta = \frac{Q_i\left[\sigma_i\,\phi\left(\frac{r_i-\mu_i}{\sigma_i}\right) - (r_i-\mu_i)\,\Phi\left(\frac{r_i-\mu_i}{\sigma_i}\right) - 2W_i^2\beta(r_i)\right]}{2\lambda_i W_i},$$

or

$$Q_i^2 = \frac{2\left[W_i\,\beta(r_i) + \theta\lambda_i\right]}{\eta C_i}, \tag{V.6}$$

and

$$\eta = \frac{\sigma_i\,\phi\left(\frac{r_i-\mu_i}{\sigma_i}\right) - (r_i-\mu_i)\,\Phi\left(\frac{r_i-\mu_i}{\sigma_i}\right)}{C_i W_i Q_i}, \tag{V.7}$$

$$\sum_{i=1}^{N} C_i\left(r_i + \frac{Q_i}{2} - \mu_i\right) = K_1, \tag{V.4}$$

and

$$\sum_{i=1}^{N} \frac{\lambda_i}{Q_i} = K_2. \tag{V.5}$$

Considering equation (V.7), the right-hand side of equation is always positive, which implies $\eta > 0$, since $\eta = 0$ implies $Q_i = \infty$ which violates equations (V.5) and (V.4).

Observing equation (V.6), the right-hand side of this equation is always positive. This suggests two possible cases for considerations.

<u>Case I</u>. $\eta > 0$, $\theta = 0$ (Ignoring the reorder workload constraint). The necessary conditions in this case are:

$$Q_i^2 = \frac{2W_i\,\beta(r_i)}{\eta C_i}, \tag{V.8}$$

$$\eta = \frac{\sigma_i \, \phi\left(\frac{r_i - \mu_i}{\sigma_i}\right) - (r_i - \mu_i)\, \Phi\left(\frac{r_i - \mu_i}{\sigma_i}\right)}{C_i W_i Q_i} \, , \qquad (V.7)$$

and

$$\sum_{i=1}^{N} C_i \left(r_i + \frac{Q_i}{2} - \mu_i \right) = K_1 \; . \qquad (V.4)$$

Case II. $\eta > 0$, $\theta > 0$ (Both constraints are active).

The necessary conditions are the same as the previous page with Eq (V.6), Eq (V.7), Eq (V.4), and Eq (V.5).

B. ITERATIVE SCHEME FOR CASE I

Equations (V.8) and (V.7) can be thought of describing two curves in the Q-r plane. Equation (V.8) clearly shows that order quantity approaches zero with increasing reorder point to infinity. Furthermore, from equation (V.8), $\frac{dQ_i}{dr_i} < 0$ for all i;

$$\frac{dQ_i}{dr_i} = \frac{-\left[\sigma_i \left(\frac{r_i - \mu_i}{\sigma_i}\right) - (r_i - \mu_i)\, \Phi\left(\frac{r_i - \mu_i}{\sigma_i}\right) \right]}{\sqrt{2 \eta C_i W_i \beta(r_i)}}$$

Taking the second partial, $\frac{d^2 Q_i}{dr_i^2}$ is greater than zero or less than zero depending upon the sign of numerator, which is

$$\frac{d^2 Q_i}{dr_i{}^2} = \frac{2 \eta C_i W_i \, \beta(r_i) \, \Phi\left(\frac{r_i - \mu_i}{\sigma_i}\right) - \left[\sigma_i \phi\left(\frac{r_i - \mu_i}{\sigma_i}\right) - (r_i - \mu_i)\, \Phi\left(\frac{r_i - \mu_i}{\sigma_i}\right) \right]^2}{\left[2 \eta C_i W_i \, \beta(r_i) \right]^{\frac{3}{2}}}$$

From equation (V.7), note that the nonnegative quantity

$$\left[\sigma_i \, \phi\left(\frac{r_i - \mu_i}{\sigma_i}\right) - (r_i - \mu_i)\, \Phi\left(\frac{r_i - \mu_i}{\sigma_i}\right) \right]$$

decreases to zero as r_i increases to infinity, at which time the order quantity will be zero. However, taking the first and second partials

from equation (V.7), $\frac{dri}{dQi} < 0$, $\frac{d^2ri}{dQ^2} > 0$ will be obtained;

$$\frac{dri}{dQi} = -\frac{\eta CiWi}{\Phi(\frac{ri-\mu i}{\sigma i})} < 0 \qquad \text{for all } i,$$

and $$\frac{d^2ri}{dQi^2} = \frac{\eta CiWi\, \sigma i}{\phi(\frac{ri-\mu i}{\sigma i})} > 0 \qquad \text{for all } i.$$

If one plots the two curves, Figure 2 will be obtained.

By setting the value of r_i to zero in Eq (V.7) and Eq (V.8), Q_A and Q_B will be obtained respectively, which are

$$Q_A = \frac{\sigma i \phi(-\frac{\mu i}{\sigma i}) - (ri - \mu i)\, \Phi(-\frac{\mu i}{\sigma i})}{\eta CiWi},$$

and

$$Q_B^2 = \frac{2Wi\left[\frac{1}{2}(\sigma i^2 + \mu i)^2 \Phi(-\frac{\mu i}{\sigma i}) + \frac{\sigma i}{2} \mu i \phi(-\frac{\mu i}{\sigma i})\right]}{\eta Ci}.$$

To have a solution, it is necessary to have $Q_A < Q_B$. Equating Eq (V.7) and Eq (V.8), yields

$$Q_i^2 = \frac{2Wi\, \beta(ri)}{\eta Ci} = \frac{\left[\sigma_i \phi(\frac{r_i - \mu_i}{\sigma_i}) - (r_i - \mu i)\, \Phi(\frac{r_i - \mu_i}{\sigma i})\right]^2}{(\eta CiWi)^2}$$

or

$$\eta = \frac{\left[\sigma i \phi(\frac{r_i - \mu_i}{\sigma_i}) - (r_i - \mu i)\, \Phi(\frac{ri - \mu i}{\sigma i})\right]^2}{2CiWi^3 \beta(r_i)}. \qquad \text{(V.9)}$$

Notice that Eq (V.9) is the function of the decision variable r only. A double iteration as a function of η, Q and r is required, i.e, iterate

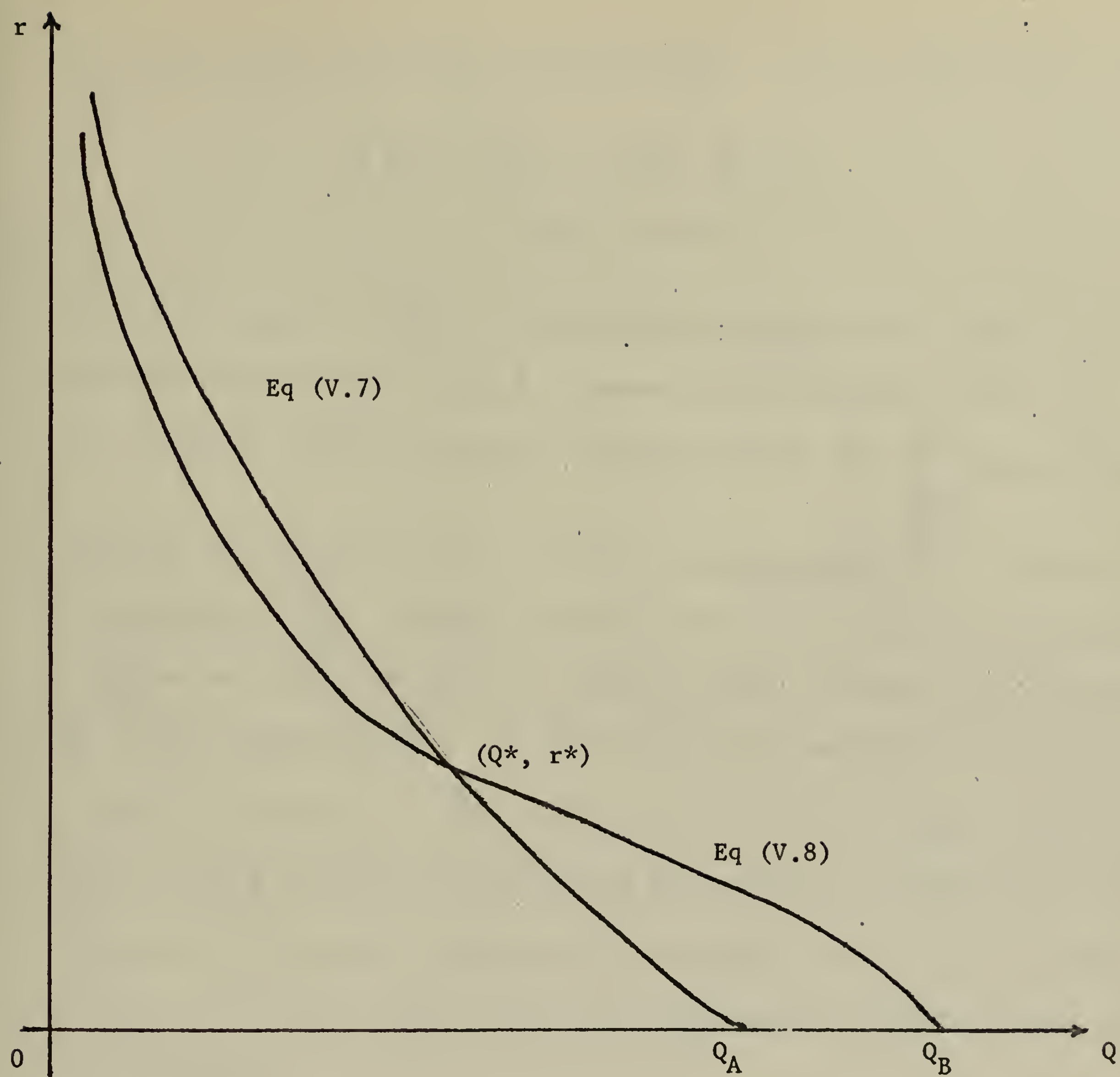

Figure 2. Q as Function of r for Case I

Q, r for a given value of the multiplier η. For the initial value of η, let

$$\delta = \min \frac{\left[\sigma_i \phi(-\frac{\mu_i}{\sigma_i}) + \mu_i \Phi(-\frac{\mu_i}{\sigma_i})\right]^2}{2C_i W_i^3 \quad (r_i=0)}$$

for all i, pick $= \frac{\delta}{2}$ as a convenient starting point. Then solve equation (V.9) for the $\{r_i\text{'s}\}$, compute $\{Q_i\text{'s}\}$ using Eq (V.8), and calculate the value of constraint using Eq (V.4). Let $\sum_{i=1}^{N} C_i(r_i + \frac{Q_i}{2} - \mu_i) = H$.

A binary search will be used. If $H > K_1$, increase η by $\frac{\delta}{4}$. If $H < K_1$, decrease η by $\frac{\delta}{4}$. Compute the value of H. If the increase (or decrease) of η has not caused the sense of inequality sign, increase (or decrease) η by the same amount $\frac{\delta}{4}$. If the sign of the inequality has changed, then reduce the increment to $\frac{\delta}{8}$, and increase (or decrease) η, solving for the $\{r_i\text{'s}\}$ and $\{Q_i\text{'s}\}$ at each value of η and computing the value of H until the sense of inequality changes again. Continue until $H = K_1$ or until H is within the tolerable region of K_1. This method will converge to the solution vector rapidly.

From the Kuhn-Tucker theorem [3], if we have a convex objective function and a convex constraint region, the necessary conditions are also sufficient. Since the first constraint under consideration is linear in r, the region is convex. To show $Z(r_i)$ is convex, it is required that the Hessian is greater than zero or equal to zero; i.e.,

$$\begin{vmatrix} \frac{\partial^2 Z_i}{\partial r_i \partial r_j} & \frac{\partial^2 Z_i}{\partial r_i \partial Q_i} \\ \frac{\partial^2 Z_i}{\partial Q_i \partial r_i} & \frac{\partial^2 Z_i}{\partial Q_i \partial Q_j} \end{vmatrix} \geq 0 .$$

Let $\alpha(r_i) = \sigma_i \phi(\frac{r_i - \mu_i}{\sigma_i}) - (r_i - \mu_i)\Phi(\frac{r_i - \mu_i}{\sigma_i})$.

Taking first and second partial with respect to each decision variable, the following will be obtained:

$$\frac{\partial Z_i}{\partial r_i} = -\frac{\alpha(r_i)}{Q_i}, \qquad \frac{Z_i}{Q_i} = -\frac{\beta(r_i)}{Q_i^2}$$

$$\frac{\partial^2 Z_i}{\partial r_i \partial r_j} = \frac{1}{Q_i}\Phi\left(\frac{r_i-\mu_i}{\sigma_i}\right), \qquad \frac{\partial^2 Z_i}{\partial Q_i \partial Q_j} = \frac{2\beta(r_i)}{Q_i^3},$$

$$\frac{\partial^2 Z_i}{\partial r_i \partial Q_i} = \frac{\alpha(r_i)}{Q_i^2}, \text{ and } \frac{\partial^2 Z_i}{\partial Q_i \partial r_i} = \frac{\alpha(r_i)}{Q_i^2}.$$

From Equation (V.10),

$$\frac{\partial^2 Z_i}{\partial r_i \partial r_j}\frac{\partial^2 Z_i}{\partial Q_i \partial Q_j} - \frac{\partial^2 Z}{\partial Q_i \partial r_i}\frac{\partial^2 Z}{\partial r_i \partial Q_i} = \frac{\Phi\left(\frac{r_i-\mu_i}{\sigma_i}\right)}{Q_i}\frac{2\beta(r_i)}{Q_i^3} - \frac{\alpha(r_i)}{Q_i^2}\frac{\alpha(r_i)}{Q_i^2},$$

or

$$\frac{1}{Q_i^4}\left[2\beta(r_i)\Phi\left(\frac{r_i-\mu_i}{\sigma_i}\right) - \left\{\alpha(r_i)\right\}^2\right]. \qquad \text{(V.11)}$$

Hence Equation (V.11) is greater than zero or less than zero depending upon the sign of bracketed quantity. Under these considerations, the convexity of Z_i is unknown; only a local minimum can be assured.

C. EXAMPLE OF THE GENERAL MODEL FOR CASE I

Once again consider the inventory of three items from IV.C, it is assumed that the distribution of lead time demand is normal with mean μ_i and variance σ_i^2 for the i^{th} item.

	Item 1	Item 2	Item 3
λ_i	1000	1500	2000
C_i	1	10	20
μ_i	100	200	300
σ_i^2	100	100	200, and let

K_1 = \$8,000 and K_2 = 15.

Reviewing section (V.A), note that the solution vectors must satisfy

$$Q_i^2 = \frac{2W_i \beta(r_i)}{\eta\, C_i} \quad , \tag{V.8}$$

$$\eta = \frac{\sigma_i \phi(\frac{r_i-\mu_i}{\sigma_i}) - (r_i-\mu_i)\Phi(\frac{r_i-\mu_i}{\sigma_i})}{C_i W_i Q_i} \tag{V.7}$$

and

$$\sum_{i=1}^{N} C_i(r_i + \frac{Q_i}{2} - \mu_i) = K_1 \ . \tag{V.4}$$

Using the search scheme described in section (V.B), this example was solved with the results shown in Table 1 utilizing three investment levels. Comparison between the General Model for Case I and Simplified Multi-item Model of section III was made for the purpose of determining how good or bad the simplified model performed in terms of time-weighted shortages and marginal cost imputed to the first multiplier. These results are shown in Tables 2 and 3. It can be seen that the Simplified Model produced time-weighted shortages which were from 25% to nearly 100% larger than the General Model for Case I.

D. ITERATIVE SCHEME FOR THE GENERAL MODEL

Again equation (V.6) and V.7) can be thought of as describing two curves in the Q - r plane. Equation (V.6) shows that the order quantity approaches a fixed value, Q_L as the reorder point goes to infinity. Furthermore, from equation (V.6), $\frac{dQ_i}{dr_i} < 0$ for all i;

$$\frac{dQ_i}{dr_i} = \frac{-\left[\sigma_i \phi(\frac{r_i-\mu_i}{\sigma_i}) - (r_i-\mu_i)\Phi(\frac{r_i-\mu_i}{\sigma_i})\right]}{\sqrt{2\eta C_i W_i\,[\beta(r_i)+\theta\lambda_i]}} \ .$$

ITEM / VARIABLE	1	2	3	REMARKS
r	340.6250	342.1875	506.2500	K_1 = 8,000
Q	60.8684	79.4553	178.4906	
η	0.0044	0.0044	0.0044	
Z_i	0.1343	1.7529	7.8755	Z = 9.7627

r	291.4063	264.8438	307.8125	K_1 = 4,000
Q	69.4359	101.1958	247.2386	
η	0.0154	0.0154	0.0154	
Z_i	0.5334	7.7740	37.9866	Z = 46.2940

r	400.0000	425.0000	687.5000	K_1 = 12,000
Q	52.6381	60.4982	137.0932	
η	0.0007	0.0007	0.0007	
Z_i	0.0194	0.2229	1.0100	Z = 1.2522

Table 2. The General Model With an Inactive Reorder Workload Constraint (Case I)

VARIABLE \ ITEM	1	2	3	REMARKS
r	234.3750	264.0625	453.1250	K_1 = 8,000
Q	746.5022	289.1190	236.0648	K_2 = 15
η	0.0055	0.0055	0.0055	
Z_i	0.2179	2.7634	10.5232	Z = 13.5045

r	175.7813	181.2500	296.8750	K_1 = 4,000
Q	746.5022	289.1190	236.0648	K_2 = 15
η	0.0173	0.0173	0.0173	
Z_i	0.8319	11.4189	42.8919	Z = 55.1428

r	300.0000	343.7500	612.5000	K_1 = 12,000
Q	746.5022	289.1190	236.0648	K_2 = 15
η	0.0011	0.0011	0.0011	
Z_i	0.0386	0.4632	1.6465	Z = 2.1483

Table 3. Simplified Multi-item Model Solutions

The second partial, $\frac{d^2Qi}{dr_i^2}$, is greater than zero or less than zero depending on the sign of the numerator;

$$\frac{d^2Qi}{dr_i^2} = \frac{2\eta CiWi\left[\beta(r_i) + \theta\lambda i\right]\Phi\left(\frac{ri-\mu i}{\sigma_i}\right) - \alpha(r_i)^2}{\left[2\eta CiWi\left\{\beta(r_i) + \theta\lambda i\right\}\right]^{\frac{3}{2}}} .$$

Observing equation (V.7), note that the same properties as section (V.B) will be obtained, which are

$$\frac{dri}{dQi} < 0, \text{ and } \frac{d^2ri}{dQi^2} > 0 \quad \text{for all i.}$$

If one plots the two curves something like Figure 3 will be obtained.

Observing Figure 3, the numerical search in Hadley and Whitin [1] will be used for the general model. To initiate the numerical procedure, it is necessary to determine reasonable multiplier values. These two multipliers are used to compute r when $Q_L = \sqrt{\frac{2\theta\lambda}{\eta C}} = Q$. The solution could be started with the point where $Q = Q_L$ on the curve defined by Eq (V.6). The r value so obtained is used in Eq (V.7) to compute a new Q value Q_2; i.e., move from the point (Q_L, r_1) on the curve defined by Eq (V.6) to a point on the curve defined by Eq (V.7) having the ordinate r_1. The Q_2 value is used in Eq (V.6) to compute a new r; i.e., we move from the curve defined by Eq (V.7) to the curve defined by Eq (V.6) at constant Q. Hence a series of steps is obtained as shown in Figure 3. It is clear that the numerical method must converge to Q* and r* for every i, given $Q_A < Q_B$.

Then compute the value of Eq (V.5) and Eq (V.4).

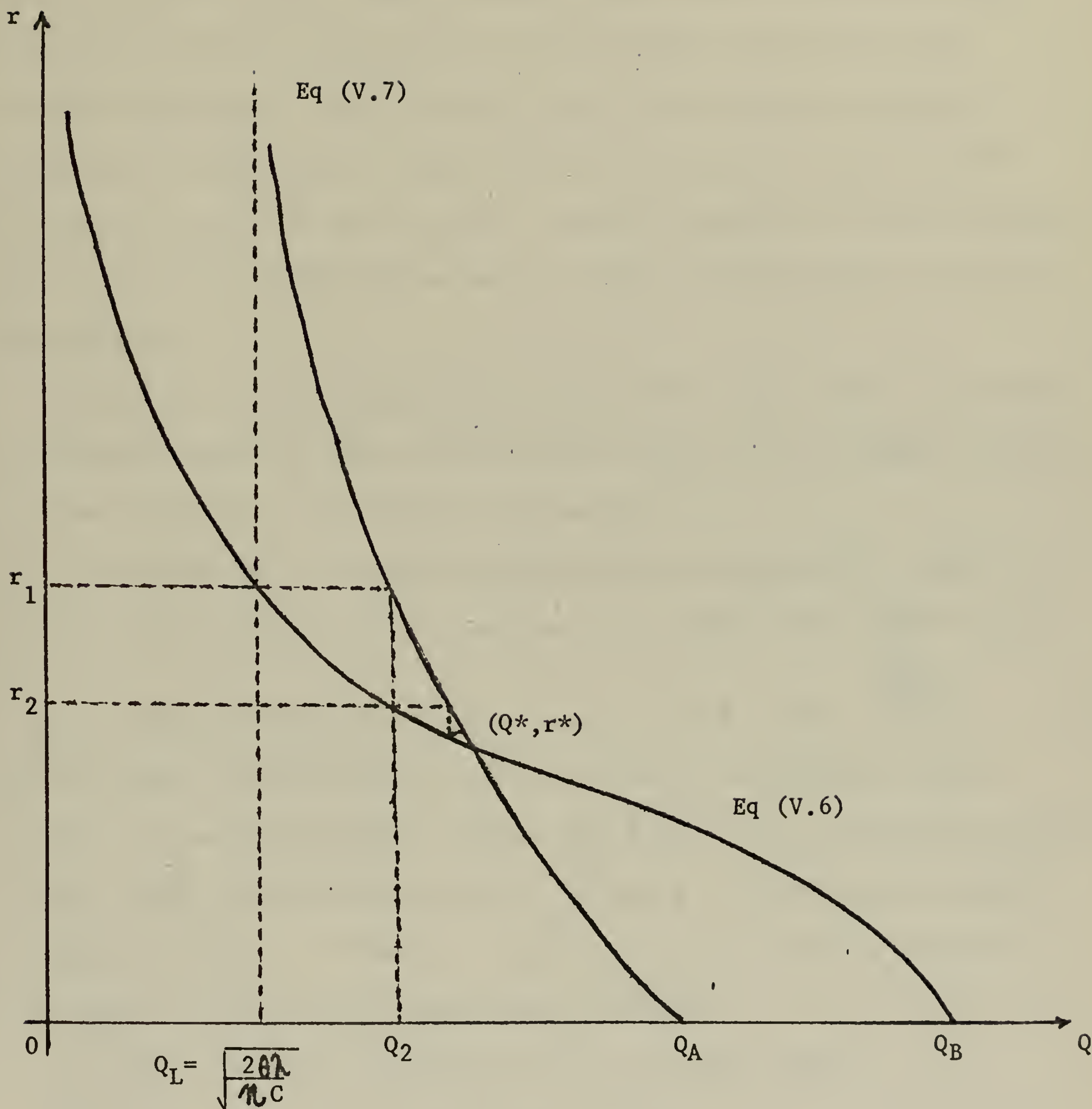

Figure 3. Q as Function of r for Case II.

In changing the values of two multipliers sequentially, η = 0.01, 0.02, ... and θ = 1,2, ..., a set of solution vectors, $\{Q, r\}$ will be obtained. Then compute the time-weighted shortages and the value of Eq (V.5) and Eq (V.4). 'Good' solution vectors can be found with small variance of Eq (V.5) and Eq (V.4) and approximate the minimum time-weighted shortages. Once obtaining this, iterate with more careful selection of multipliers, compute the objective function and check the variances of Eq (V.5) and Eq (V.4). Continue iterating on the multipliers until tolerable differences between the right and left sides of Eq (V.5) and Eq (V.4).

More sophisticated approaches are available. Among these is a search technique proposed by Fiacco and McCormick [4] , and a two state variables optimal allocation using dynamic programming.

The sequential unconstrained minimization technique [4] , SUMT, is based on the minimization of a new function $P(X,\rho) = f(X) + \rho \sum_{i=1}^{N} 1/g_i(X)$ over a strictly monotonic decreasing sequence of ρ- values $\{\rho_k\}$. Under certain restrictions that will be reviewed subsequently, there exists a sequence of feasible points $\{X(\rho_k)\}$ that respectively minimize $\{P(X, \rho_k)\}$, and it follows that $X(\rho_k) \rightarrow \bar{X}$, a solution of original function as $\rho_k \rightarrow 0$ $(k \rightarrow \infty)$. The following is a concise summary of the steps describing the computational algorithm:

1. Select a point X^o interior to the feasible region.
2. Select ρ_1, the initial value of ρ using

$$\rho_1 = - \nabla f(X^o)^T \nabla p(X^o) / |\nabla p(X^o)|^2 ,$$

or

$$\rho_1 = \left[\frac{\nabla f(X^o)^T H_2^{-1} \nabla f(X^o)}{\nabla p(X^o)^T H_2^{-1} \nabla p(X^o)} \right]^{\frac{1}{2}},$$

where $p(X) = \sum_i 1/g_i(X)$, H_1 and H_2 are the Hessians of $f(X)$ and $p(X)$ respectively.

3. Determine the minimum of $P(X,\rho_k)$ for the current value of ρ_k using Gradient Methods,

$$X^2 = X^1 - \theta \nabla P\,(X^1),$$

or

$$X^2 = X^1 - \theta\left[\partial^2 P(X^1) / \partial X_i \partial X_j\right]^{-1} \nabla P(X^1)$$

4. If $k > 1$, estimate solution using extrapolation formula.

5. Terminate computations if final convergence criteria are satisfied, $f(\bar{X}) - G\left[X(\rho), \mu(\rho)\right] < \varepsilon$. The theoretical optimum value v_o is bounded by the dual and primal function values respectively,

$$G\left[X(\rho), \mu(\rho)\right] \leq v_o \leq f\left[X(\rho)\right].$$

If the value v_o is not within the bounds above, go to step (6).

6. Select $\rho_{k+1} = \rho_k / C$, where $C > 1$.

7. If $k > 1$, estimate minimum for reduced ρ - value, using an extrapolation formula.

E. EXAMPLE OF THE GENERAL MODEL (CASE II)

Consider an inventory of three items with the following characteristics:

	Item 1	Item 2	Item 3
λ_i	1000	1500	2000
Ci	1	10	20
μ_i	100	200	300
σ_i^2	100	100	200

K_1 = \$8,000 $\qquad$ K_2 = 15

The search technique proposed by Fiacco and McCormick [4] was used. The following initial feasible solution was used: $Q_1 = 600$, $r_1 = 200$, $Q2 = 270$, $r_2 = 260$, $Q_3 = 300$, $r_3 = 400$. $Zi = \sum_{i=1}^{3} \frac{\beta(ri)}{Qi} = 17.8084$ and $\rho_1 = -1.6991$ were obtained.

This example was solved with the following results,

$Q_1 = 362.718$, $r_1 = 322.1433$,

$Q_2 = 266.403$, $r_2 = 272.0930$,

$Q_3 = 302.6870$, $r_3 = 425.7958$, $\rho_1 = -0.00002231$,

and the value of the objective function (Z) was Z = 13.39781.

VI. SUMMARY AND CONCLUSIONS

It is apparent that the general models proposed do not represent the ultimate answer in multi-item inventory theory. Since the conditions of concavity and convex sets are not maintained for the general model, the computational algorithms which are based on concavity and convex sets of objective and constraing respectively could not apply directly. Though the P function in section (V.D) appears prohibitively difficult to work with computationally, it has been understood that it is minimized efficiently and accurately for the great preponderance of problems solved to date, by means of the second order optimum gradient method.

From the example problem we see the general problem in section (V.E) provides solution which is better than the solution of the simplified continuous review model in section (IV.C). The major advantage of the simplified continuous review model is its computational ease.

While an efficient algorithm for solution of the general problem has not been presented, the advent of high speed computer has opened this field of numerical iterative procedures for large inventory systems.

Since the value of the constraints are more easily determined than the order cost and holding cost, the model proposed seems much more appropriate than the traditional variable cost minimization models.

From the example problem given in section (III), it is apparent that time-weighted shortages and expected number of units short per unit time seem to follow each other closely.

Finally, all the solutions to example problems were obtained by utilizing a digital computer.

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(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Naval Postgraduate School Monterey, California 93940	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

A Goal-Constraint Formulation for Multi-Item Inventory Systems

4. DESCRIPTIVE NOTES (Type of report and, inclusive dates)
Master's Thesis; September 1970

5. AUTHOR(S) (First name, middle initial, last name)

Ui Chong Choe

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
September 1970	45	4

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)
b. PROJECT NO.	
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d.	

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13. ABSTRACT

Variations on the traditional cost minimization of continuous review formulation are investigated in an effort to improve service as measured in terms of time-weighted shortages per unit time. It is proposed that the minimization of time-weighted shortages per unit time will improve service in current Navy Supply Operations. Various models are presented, without reliance upon unknown parameters such as order cost and carrying cost, with necessary conditions and solution algorithms.

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Continuous Review Inventory Policies						
Multi-Item Inventory Control						

Thesis 121589
C44883 Choe
c.1 A goal-constraint
formulation for multi-
item inventory sys-
tems.